MECHATRONIC DESIGN

Lab 1: Introduction to Finite Difference (FD) method

The goal of the first laboratory is to approximate solutions to differential equations, i.e., to find a function (or some discrete approximation to this function) that satisfies a given relationship between various of its derivatives on some given region of space and/or time, along with some boundary conditions along the edges of this domain. In general, this is a difficult problem, and only rarely can an analytic formula be found for the solution. **A finite difference method proceeds by replacing the derivatives in the differential equations with finite difference approximations.** This gives a large but finite algebraic system of equations to be solved in place of the differential equation, something that can be done on a computer.

Before tackling this problem, we first consider the more basic question of how we can approximate the derivatives of a known function by finite difference formulas based only on values of the function itself at discrete points. Besides providing a basis for the later development of finite difference methods for solving differential equations, this allows us to investigate several key concepts such as the *order of accuracy* of an approximation in the simplest possible setting.

Let $u(x)$ represent a function of one variable that, unless otherwise stated, will always be assumed to be smooth, meaning that we can differentiate the function several times and each derivative is a well-defined bounded function over an interval containing a particular point of interest $\bar{x}$.

Suppose we want to approximate $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ by a finite difference approximation based only on values of u at a finite number of points near $\bar{x}$. One obvious choice would be to use

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}}_{+} \approx \frac{\boldsymbol{u(\bar{x}+\Delta x)-u(\bar{x})}}{\boldsymbol{\Delta x}} \tag{1}$$

for some small value of *Δx*. Note that $\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}}_{+}$ is the slope of the line interpolating *u* at the points $\bar{x}$ and $\bar{x}+\Delta x$ (see Figure 1.)

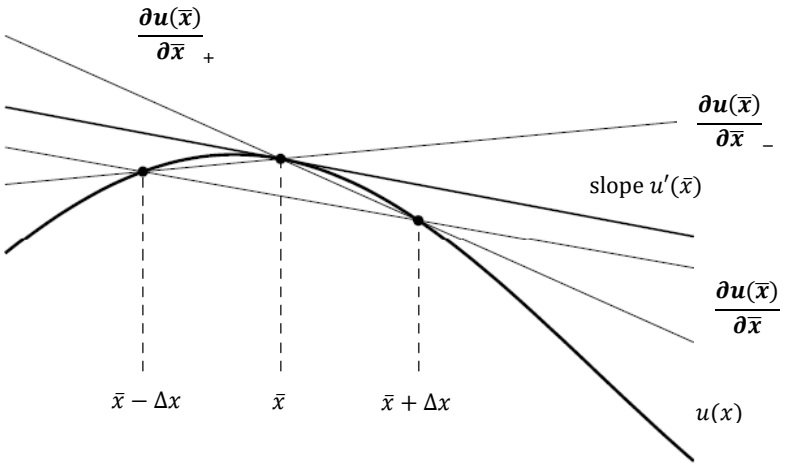

Fig. 1. Various approximations to $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ interpreted as the slope of secant lines.

Equation (1) is a **one-sided approximation** to $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ since *u(x)* is evaluated only at values of $x \geq \bar{x}$. Another one-sided approximation would be

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}}_{-} \approx \frac{\boldsymbol{u(\bar{x})-u(\bar{x}-\Delta x)}}{\boldsymbol{\Delta x}}. \tag{2}$$

Each of these formulas gives a *first order accurate* approximation to $u'(\bar{x})$, meaning that the size of the error is roughly proportional to Δx itself. Another possibility is to use the ***centered approximation***

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}} \approx \frac{\boldsymbol{u(\bar{x}+\Delta x)-u(\bar{x}-\Delta x)}}{\boldsymbol{2\Delta x}} \tag{3}$$

Equation (3) is the slope of the line interpolating u at $\bar{x} - \Delta x$ and $\bar{x} + \Delta x$ and is simply the average of the two one-sided approximations defined above using Eqs. (1) and (2). **From Figure 1 it should be clear that we would expect $\frac{\partial u(\bar{x})}{\partial \bar{x}}$ to give a better approximation than either of the one-sided approximations.** In fact, this gives a second order accurate approximation - the error is proportional to Δx^2 and hence is much smaller than the error in a first order approximation when Δx is small ($\Delta x < 1$).

How are the above finite difference approximations and truncation errors analyzed? The standard approach is to expand each of the function values of u in a Taylor series about the point $\bar{x}$, e.g.,

$$u(\bar{x} + \Delta x) = u(\bar{x}) + \Delta x \cdot \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{2}\Delta x^2 \cdot \frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} + \frac{1}{6}\Delta x^3 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3} + O(\Delta x^4) \tag{4}$$

$$u(\bar{x} - \Delta x) = u(\bar{x}) - \Delta x \cdot \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{2}\Delta x^2 \cdot \frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} - \frac{1}{6}\Delta x^3 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3} + O(\Delta x^4) \tag{5}$$

Using Equation (4) allows us to compute that

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}}_{+} \approx \frac{\boldsymbol{u(\bar{x} + \Delta x) - u(\bar{x})}}{\boldsymbol{\Delta x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{2}\Delta x \cdot \frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} + \frac{1}{6}\Delta x^2 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3} + O(\Delta x^3) \tag{6}$$

Recall that $\bar{x}$ is a fixed point so that $\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$, $\frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}$, etc., are fixed constants independent of Δx. They depend on u of course, but the function is also fixed as we vary Δx.

For Δx sufficiently small, the error will be dominated by the first term $\frac{1}{2}\Delta x \cdot \frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$ and all the other terms will be negligible compared to this term, so we expect the error to behave roughly like a constant times Δx, where the constant has the value $\frac{1}{2}\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2}$.

Similarly, from Equation (5) we can compute that the error in $D_{-}u(\bar{x})$ is

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}}_{-} \approx \frac{\boldsymbol{u(\bar{x}) - u(\bar{x} - \Delta x)}}{\boldsymbol{\Delta x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{2}\Delta x \cdot \frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} - \frac{1}{6}\Delta x^2 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3} + O(\Delta x^3). \tag{7}$$

Subtracting Equation (5) from (4) gives

$$u(\bar{x} + \Delta x) - u(\bar{x} - \Delta x) = 2\Delta x \cdot \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{3}\Delta x^3 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3}(\bar{x}) + O(\Delta x^5) \tag{8}$$

so the central difference approximation is

$$\frac{\boldsymbol{\partial u(\bar{x})}}{\boldsymbol{\partial \bar{x}}} \approx \frac{\boldsymbol{u(\bar{x} + \Delta x) - u(\bar{x} - \Delta x)}}{\boldsymbol{2\Delta x}} \approx \frac{\partial u(\bar{x})}{\partial \bar{x}} + \frac{1}{6}\Delta x^2 \cdot \frac{\partial^3 u(\bar{x})}{\partial \bar{x}^3} + O(\Delta x^4) \tag{9}$$

This confirms the second order accuracy of this approximation. Note that all the odd order terms drop out of the Taylor series expansion (9) for $\frac{\partial u(\bar{x})}{\partial \bar{x}}$. This is typical with centered approximations and typically leads to a higher order approximation.

Approximations to the second order derivative $\frac{\partial^2 u(\bar{x})}{\partial^2 \bar{x}}$ can be obtained in an analogous manner (using Taylor series) leading to

$$\frac{\partial^2 u(\bar{x})}{\partial \bar{x}^2} \approx \frac{\boldsymbol{u(\bar{x} + \Delta x) - 2u(\bar{x}) + u(\bar{x} - \Delta x)}}{\boldsymbol{\Delta x^2}} + O(\Delta x^2) \tag{10}$$

The second order mixed derivative $\frac{\partial^2 u(\bar{x},\bar{y})}{\partial \bar{x}\partial \bar{y}}$ is:

$$\frac{\partial^2 u(\bar{x},\bar{y})}{\partial \bar{x}\partial \bar{y}} \approx \frac{u(\bar{x} + \Delta x, \bar{y} + \Delta y) - u(\bar{x} - \Delta x, \bar{y} + \Delta y) - u(\bar{x} + \Delta x, \bar{y} - \Delta y) + u(\bar{x} - \Delta x, \bar{y} - \Delta y)}{4\Delta x \Delta y} + O(\Delta x^2, \Delta y^2 \tag{11}$$

Tasks to do:

Task 1:

Solve a one-dimensional acoustic wave equation using finite difference method for an **aluminum/copper/steel** rod. Implement a source code in MATLAB to obtain wave displacement in time and space domains. The one-dimensional wave equation is:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad c = \sqrt{\frac{E}{\rho}}, \qquad 0 \le x \le L, \qquad t \ge 0 \tag{12}$$

where E is Young's modulus, ρ is mass density, L is the length of the rod.

Use the following constant parameters for your calculations: **Δx = 1 mm, Δt = 0.12 µs**. The excitation signal is a 200 kHz two cycle sine wave modulated by a Hanning window as shown in Fig. 2.

Calculate the wave speed of the simulated wave motion.

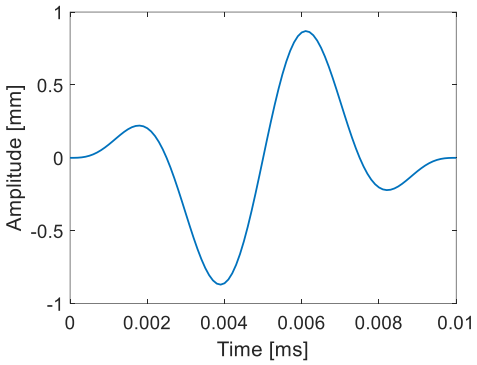

Fig. 2. The excitation signal has a frequency of 200 kHz and is a two-cycle sine wave. The signal is also modulated by a Hanning window.

Exemplary results:

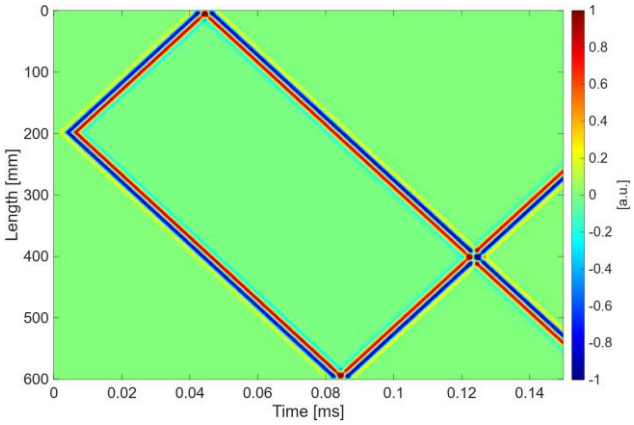

Fig. 3. A spatio-temporal acoustic wave motion profile measured with the excitation located at a distance of 200 mm.

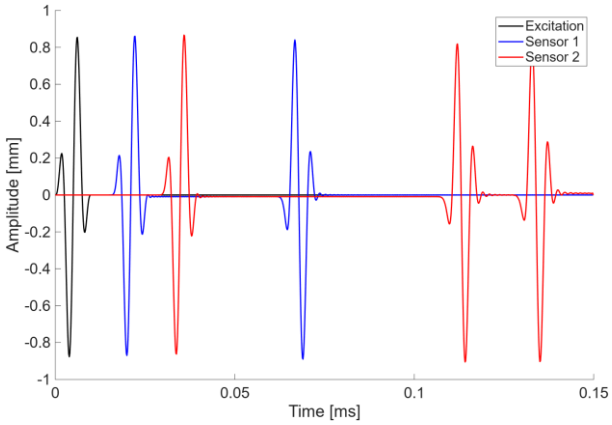

Fig. 4. Temporal acoustic wave motion as measured at 120 mm and 350 mm for sensors 1 and 2, respectively.

Task 2:

Solve a two-dimensional acoustic wave equation using finite difference method for an **aluminum** plate. Implement a source code in MATLAB to obtain wave displacement in time and space domains. The two-dimensional acoustic wave equation is:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right),$$
$$c = \sqrt{\frac{E}{\rho}}, \qquad 0 \le x \le L_1, \qquad 0 \le y \le L_2, \qquad t \ge 0 \tag{13}$$

where E is Young's modulus, ρ mass density, L is the length of the rod. Use the following constant parameters for your calculations: **Δx = Δy = 1 mm, Δt = 0.12 μs**.

Use the excitation signal from Task 1.

Exemplary results:

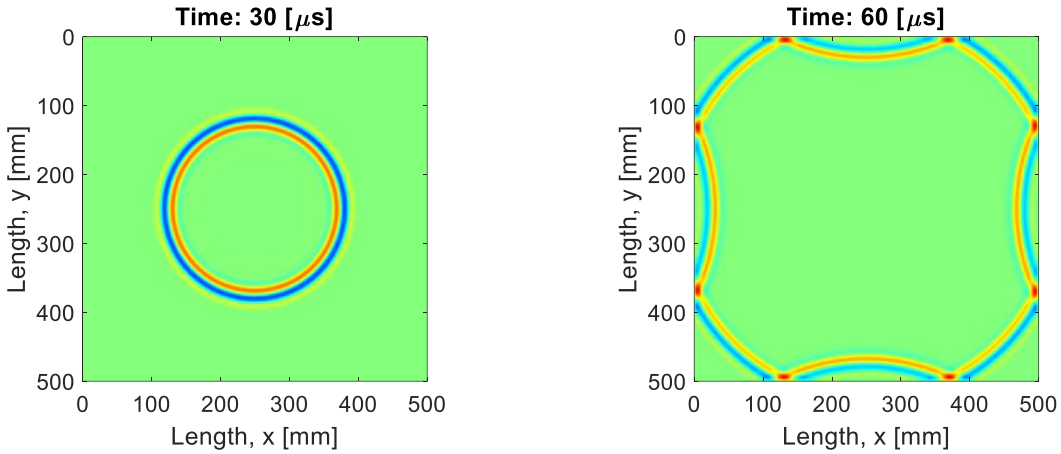

Fig. 5. Wave motion data for the excitation located in the middle of the plate.

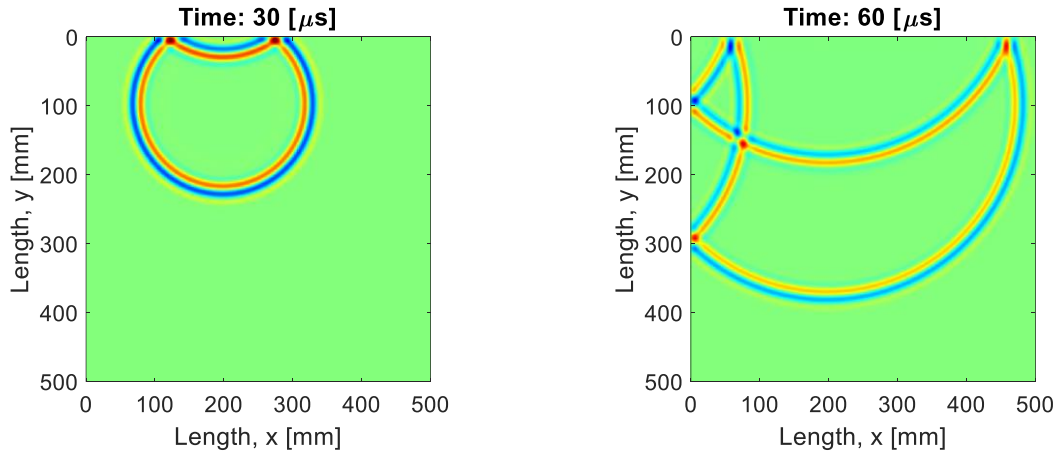

Fig. 6. Wave motion data for the excitation located in at the location x = 200 mm, y = 100 mm.

Task 3:

Do the simulations again from Task 2, but this time add a discontinuity to your model. For example, you could do it like shown in Fig. 6.

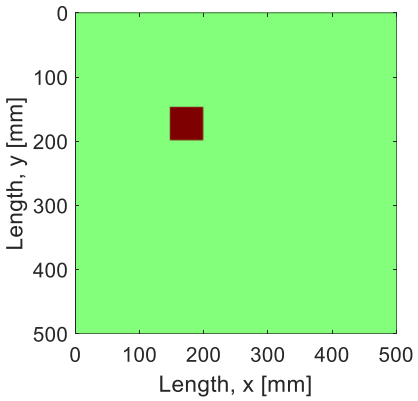

Fig. 7. Visualization of a 2D model with a square-shaped discontinuity.

Exemplary results:

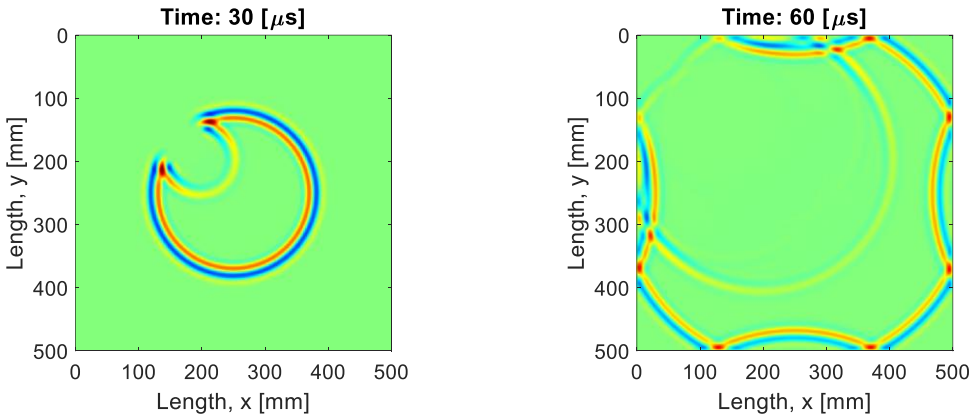

Fig. 8. Wave motion data for the excitation located in the middle of the plate.

Task 4:

Solve two-dimensional elastodynamic wave equations for an isotropic, linear material using finite difference method for an aluminum plate. Implement a source code in MATLAB to obtain wave displacement in time and space domains.

$$\begin{cases} \rho \dfrac{\partial^2 u_x}{\partial t^2} = \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xz}}{\partial z} + F_x \\ \rho \dfrac{\partial^2 u_z}{\partial t^2} = \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{zz}}{\partial z} + F_z \end{cases}$$

whereas stress-strain relations are as follow

$$\tau_{xx} = (\lambda + 2\mu)\frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_z}{\partial z}$$

$$\tau_{zz} = (\lambda + 2\mu)\frac{\partial u_z}{\partial z} + \lambda \frac{\partial u_x}{\partial x}$$

$$\tau_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

where
ρ density,
μ shear modulus,
λ first Lamé constant
u displacement vector
F force vector

Exemplary results:

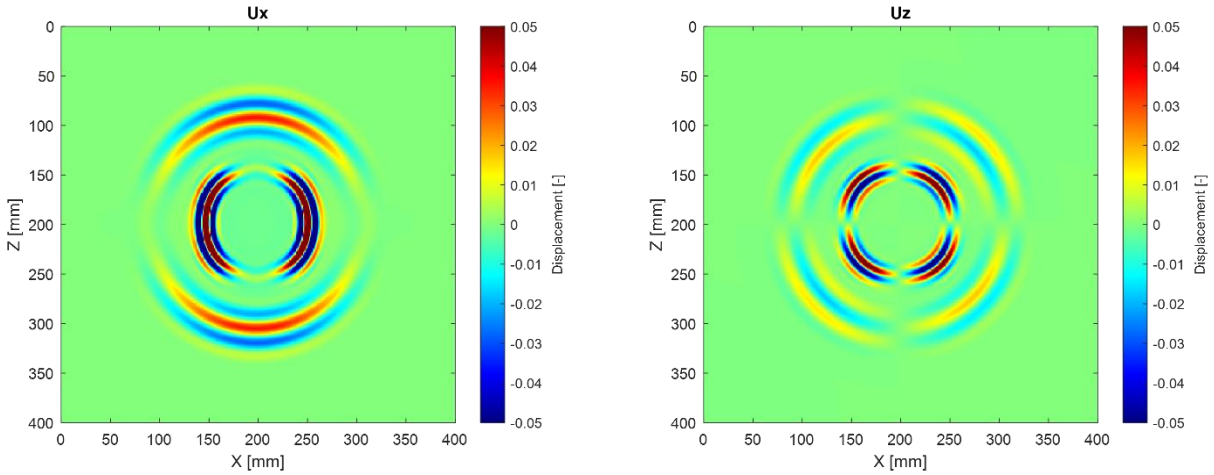

Fig. 9. Wave motion data for Ux and Uz.